

Analytical method of spectra calculations in the Bargmann representation

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We formulate a universal method for solving an arbitrary quantum system which, in the Bargmann representation, is described by a system of linear equations with one independent variable, such as one- and multi-photon Rabi models, or N level systems interacting with a single mode of the electromagnetic field and their various generalizations. We explain three types of conditions that determine the spectrum and show their usage for two deformations of the Rabi model. We prove that the spectra of both models are just zeros of transcendental functions, which in one case are given explicitly in terms of confluent Heun functions.

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INTRODUCTION

Our goal and result is a general method which allows to properly determine eigenvalues and eigenfunctions for a wide class of quantum systems. It is adequate for quantum optical setting where the Bargmann representation allows for natural parametrization of the electromagnetic degree of freedom and the resulting differential equations are ordinary and linear. We then show its application to two systems, which are generalizations of the famous Rabi model characterized by the Hamiltonian

$$H = a^\dagger a + \mu \sigma_z + \lambda \sigma_x (a^\dagger + a), \quad (1)$$

where a , a^\dagger are the photon annihilation and creation operators, μ , λ are the level separation and photon-atom coupling constant, and σ_x , σ_z are the Pauli spin matrices.

This fundamental system describes interaction of a two-level atom with a single harmonic mode of the electromagnetic field. Originally, it was introduced to describe the effect of a rapidly varying, weak magnetic field on an oriented atom possessing nuclear spin [1]. It has been recently applied to a great variety of physical systems, including cavity and circuit quantum electrodynamics, quantum dots, polaronic physics and trapped ions, see [2–7].

Usually coupling between “natural” two-level atoms and the single bosonic mode of radiation is quite weak and the rotating wave approximation is valid. It leads to a solvable, the so-called Jaynes-Cummings, model. However, recent achievements in circuit quantum electrodynamics have enabled the exploration of such regimes, e.g., the ultra-strong and the deep strong coupling regimes of light-atom interaction so that the Jaynes-Cummings model begins to fail. Effects of counter-rotating terms cannot be more

neglected and terms containing simultaneous excitation or deexcitation of both the atom and the field must be taken into account [2–6]. The second reason for its recent renaissance is the realization that the strong coupling regimes might require more interaction terms than just those mentioned above. One such generalization, the so-called Rabi model with broken symmetry, was proposed in [7] and its additional term was justified physically as spontaneous emission by the atom. The Hamiltonian of this generalization is

$$H_\varepsilon = a^\dagger a + \mu \sigma_z + \lambda \sigma_x (a^\dagger + a) + \varepsilon \sigma_x. \quad (2)$$

This will be the first example we study. The second one was proposed in [8, 9]. It includes a nonlinear coupling term between the atom and the cavity:

$$H = \left(\omega + \frac{U}{2} \sigma_z \right) a^\dagger a + \frac{\omega_0}{2} \sigma_z + g \sigma_x (a^\dagger + a). \quad (3)$$

An alternative physical motivation of the additional term is that it could arise in the dispersive limit of the Jaynes-Cummings model. However, the first possibility is more accessible experimentally as described in [8]. We chose to keep the notation of that paper, for the second model, to facilitate comparison. A quick inspection shows that the parameters of models (2) and (3) are related via

$$\omega = 1, \quad \omega_0 = 2\mu, \quad g = \lambda. \quad (4)$$

Although the spectrum of the classical Rabi model has been determined by numerical and approximate methods before, see, e.g., [10–13], there still is a lack of a general approach which works well for arbitrary parameters values and which has a solid mathematical foundations. Recently several approaches devoted to determination of

the spectrum of this and similar models have appeared, see, e.g., [7, 14–17], and references therein. The authors have also applied the present method as outlined in the preliminary preprint [18], to determine the full spectrum in [19], including some isolated points that are usually overlooked.

It should be underlined that the Rabi model is one of the simplest ones in quantum physics. This is why the knowledge of its exact eigenvalues and eigenfunctions is of great theoretical importance. Although the question about the spectrum and eigenstates comes from physics, it is a mathematical one. It is obvious that unjustified methods may lead to incorrect physical interpretations of considered models.

In the Bargmann-Fock representation, see [20], the two-component wave function $\psi = (\psi_1, \psi_2)$ is an element of Hilbert space $\mathcal{H}^2 = \mathcal{H} \times \mathcal{H}$, where \mathcal{H} is the Bargmann-Fock Hilbert space of entire functions of one variable $z \in \mathbb{C}$. The elegant connection with the standard picture is that the annihilation and creation operators a , and a^\dagger become ∂_z and multiplication by z , respectively, for clearly $[\partial_z, z] = 1$. The scalar product in \mathcal{H} is given by

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathbb{C}} \overline{f(z)} g(z) e^{-|z|^2} d(\operatorname{Re}(z)) d(\operatorname{Im}(z)).$$

It is worth mentioning that this space was also introduced, independently of Bargmann, by J. Newman and H. S. Shapiro [21, 22]. However their motivation was connected with works of Ernst Fischer [23, 24]. They tried to generalize a very beautiful construction of E. Fisher valid for polynomials.

The Hilbert space \mathcal{H} has several peculiar properties. Let us mention two of them:

1. $f(z) \in \mathcal{H}$ does not imply that $f'(z) \in \mathcal{H}$.
2. $f(z) \in \mathcal{H}$ does not imply that $zf(z) \in \mathcal{H}$.

To understand these rather strange properties we have to recall some definitions and facts from the theory of entire functions, see [25, 26]. If $f(z)$ is an entire function, then to characterize its growth, the following function is used:

$$M_f(r) := \max_{|z|=r} |f(z)|. \quad (5)$$

We omit the subscript f later on, because the investigated function is known from the context. If for an entire function $f(z)$ we have

$$\limsup_{r \rightarrow \infty} \frac{\ln(\ln M(r))}{\ln r} = \varrho, \quad \text{with } 0 \leq \varrho \leq \infty, \quad (6)$$

then ϱ is called the order (or growth order) of $f(z)$. If, further, the function has positive order $\varrho < \infty$ and satisfies

$$\limsup_{r \rightarrow \infty} \frac{\ln M(r)}{r^\varrho} = \sigma, \quad (7)$$

then we say that $f(z)$ is of order ϱ and of type σ .

Assume that $f(z)$ belongs to \mathcal{H} , then one can prove the following facts [20]:

1. $f(z)$ is of order $\varrho \leq 2$.
2. If $\varrho = 2$, then $f(z)$ is of type $\sigma \leq \frac{1}{2}$.

If $\varrho = 2$ and $\sigma = \frac{1}{2}$, then the question whether $f(z) \in \mathcal{H}$ requires a separate investigation. Particularly in the mentioned case when $f(z) \in \mathcal{H}$ but $f'(z) \notin \mathcal{H}$ the function is of order $\varrho = 2$ and type $\sigma = \frac{1}{2}$. For additional details see [27].

The usefulness of this representation can immediately be seen with the harmonic oscillator, which represents the radiation. The time-independent Schrödinger equation for energy E is simply $H\psi(z) = z\psi'(z) = E\psi(z)$ and one immediately recovers the orthonormal eigenbasis as $\{z^n/\sqrt{n!}\}_{n \in \mathbb{N}}$. The connection with the usual space of square-integrable functions of q is given by the integral kernel $\exp(-(z^2 + q^2)/2 + \sqrt{2}qz)$ which is one of the forms of the generating function for the Hermite polynomials. Each z^n thus corresponds to the appropriately normalized wave function $e^{-q^2/2} H_n(q)$. In this basis the operator a is just an infinite matrix with entries on the superdiagonal, so all the mentioned Hamiltonians can be constructed as tensor products of such matrices with the sigma matrices. This allows for direct numerical diagonalization. However, the open question that we wish to tackle is how to determine the spectrum rigorously with as explicit exact formulas as possible.

In the Bargmann-Fock representation energy E belongs to the spectrum of the problem, if and only if, for this value of E the equation $H\psi = E\psi$ has entire solution $\psi = (\psi_1, \psi_2) \in \mathcal{H}^2$. We want to find, if possible, a formula for those values of E .

As we already mentioned, in the Bargmann-Fock representation, the considered models are described by a system of linear differential equations. We shall see that the equations in question will involve regular singular points and a possibly irregular point at infinity on the complex z plane. The conditions that the considered system has a solution with components belonging to \mathcal{H} , are roughly threefold:

- Local conditions. At each regular singular point $z = s$ there exists at least one solution which is holomorphic on an open set containing s .
- Global conditions. Among all solutions which are locally holomorphic, we can find at least one at each singular point such that they are a holomorphic continuation of one another.
- Normalization conditions. The entire function obtained in the above way must have finite Bargmann norm.

Our method gives straightforward and natural compatibility conditions in term of Wronskian determinants and was first formulated in our unpublished preprint [18]. For simplicity sake we chose the two models that can be given either as a system of two first order equations or one equation of the second order. The application to higher order equations, as those investigated in [28] or [29], will appear in future work [30].

In the Bargmann representation, the first considered model is described by the following system of two differential equations

$$\begin{aligned} (z + \lambda) \frac{d\psi_1}{dz} &= (E - \varepsilon - \lambda z)\psi_1 - \mu\psi_2, \\ (z - \lambda) \frac{d\psi_2}{dz} &= (E + \varepsilon + \lambda z)\psi_2 - \mu\psi_1. \end{aligned} \quad (8)$$

We will use this model to illustrate the single equation approach below. The second system takes the form

$$\begin{aligned} \left(\omega + \frac{U}{2}\right) z\psi'_1 + \frac{\omega_0}{2}\psi_1 + g\psi'_2 + gz\psi_2 &= E\psi_1, \\ \left(\omega - \frac{U}{2}\right) z\psi'_2 - \frac{\omega_0}{2}\psi_2 + g\psi'_1 + gz\psi_1 &= E\psi_2. \end{aligned} \quad (9)$$

When we change the independent variable

$$z \rightarrow y = \frac{\sqrt{4\omega^2 - U^2}}{2g} z, \quad (10)$$

then this system can be rewritten in the matrix form

$$\frac{d}{dy}\psi = \mathbf{A}(y)\psi, \quad (11)$$

where matrix \mathbf{A} has the following entries

$$\begin{aligned} a_{11} &= \frac{y(-4g^2 + (U - 2\omega)(2E - \omega_0))}{(U^2 - 4\omega^2)(y^2 - 1)}, \\ a_{12} &= -\frac{4g^2y^2 + (U + 2\omega)(2E + \omega_0)}{(U + 2\omega)\sqrt{-U^2 + 4\omega^2}(y^2 - 1)}, \\ a_{21} &= \frac{4g^2y^2 - (U - 2\omega)(2E - \omega_0)}{(U - 2\omega)\sqrt{-U^2 + 4\omega^2}(y^2 - 1)}, \\ a_{22} &= -\frac{y(4g^2 + (U + 2\omega)(2E + \omega_0))}{(U^2 - 4\omega^2)(y^2 - 1)}. \end{aligned} \quad (12)$$

In the above one must assume $4\omega^2 \neq U^2$, and if that is not the case the type and position of singularities changes considerably. This special case requires the aforementioned closer investigation of the behaviour at infinity which we will present in our future work [31].

Both our examples share the same features. They are described by linear systems with rational coefficients and in both cases the problem is to distinguish those parameters values for which the system admits a solution in \mathcal{H} . As is well known, a system of linear equations can

be reduced to one equation of higher order. For example, an elimination of $\psi_2(z)$ from (8) gives one second order equation for $\varphi(z) := \psi_1(z)$ of the form

$$\varphi'' + p(z)\varphi' + q(z)\varphi = 0, \quad (13)$$

with

$$\begin{aligned} p(z) &= -\frac{\lambda + 2\varepsilon\lambda + z(2E - 1 + 2\lambda^2)}{z^2 - \lambda^2}, \\ q(z) &= -\frac{\varepsilon^2 - E^2 + 2z\varepsilon\lambda + \lambda(\lambda + z(z\lambda - 1)) + \mu^2}{z^2 - \lambda^2}. \end{aligned} \quad (14)$$

METHOD

First, we make some general remarks. Although an arbitrary system of n linear differential equations with rational coefficients of the first order can be transformed into a single linear equation, in some cases it seems natural to work directly with the given system. However, for a system to decide which singular point is regular, and what the exponents are at this point is not so obvious. All known algorithms which allow to determine these basic characteristics of singular points for a system are rather involved. Still, the general procedure described below can easily be adapted to a system, only some technical points are more intricate [32].

We make the following assumptions. The considered equation has the form

$$\varphi'' + p(z)\varphi' + q(z)\varphi = 0, \quad (15)$$

where $p(z)$ and $q(z)$ are rational functions. All singular points $\Sigma := \{s_1, \dots, s_m\} \subset \mathbb{C}$ are regular and they are poles of the equation's coefficients. Thus, all poles of $p(z)$ are of order not greater than 1, and all poles of $q(z)$ are of order not greater than 2. The infinity is an irregular singular point (in all examples of quantum optics we know it seems to be the case). Thus either

$$P(\zeta) = \frac{2}{\zeta} - \frac{1}{\zeta^2}p\left(\frac{1}{\zeta}\right) \quad (16)$$

has a pole of order greater than 1 at $\zeta = 0$, or

$$Q(\zeta) := \frac{1}{\zeta^4}q\left(\frac{1}{\zeta}\right) \quad (17)$$

has a pole of order greater than 2 at $\zeta = 0$.

We also assume that the singular points are numbered and located in such a way that there exist open disks $D(s_i, r_i)$ centred at s_i and of radius r_i satisfying the following conditions:

1. $\Sigma \cap D(s_i, r_i) = \{s_i\}$ for $1 \leq i \leq m$.
2. $U_i := D(s_i, r_i) \cap D(s_{i+1}, r_{i+1}) \neq \emptyset$ for $1 \leq i < m$.

Remark 1. As a matter of fact our method works for an arbitrary configuration of singular points. However for such general configuration one has to use the continuation theory, see, e.g., [33, Ch. IX].

We can now return to the description of our method itself. Assume that $\varphi(z)$ is an entire solution of (15). If $z_0 \notin \Sigma$, then all solutions of (15) are locally holomorphic in a neighbourhood of z_0 . So, local conditions do not give any restrictions on the parameters. At a singular point $s_i \in \Sigma$ the form of local solutions depends on exponents at this point. We denote them by ρ_i and ς_i . A general basis of solutions is

$$\begin{aligned}\phi_{i1} &= (z - s_i)^{\rho_i} h_i(z), \\ \phi_{i2} &= l_i \phi_{i1} \ln(z - s_i) + (z - s_i)^{\varsigma_i} k_i(z),\end{aligned}\quad (18)$$

where $h_i(z)$ and $k_i(z)$ do not vanish at $z = s_i$ and are locally holomorphic with radius of convergence not smaller than the distance from s_i to the closest other singular point. The logarithmic term might arise only when the difference of exponents is an integer, although this condition is not sufficient. By assumption, the power expansion of $\varphi(z)$ at s_i is holomorphic. This implies that at least one exponent, let us say ρ_i must be a non-negative integer, i.e., $\rho_i \in \mathbb{N} = \{0, 1, \dots\}$ for all i . In other words, ρ_i is either the largest integer exponent or the only integer exponent. Hence, at each singular point we have a locally holomorphic solution $\varphi_i(z)$ of the form

$$\varphi_i(z) = \zeta_i (z - s_i)^{\rho_i} h_i(z) + \xi_i (z - s_i)^{\varsigma_i} k_i(z), \quad (19)$$

and ξ_i is nonzero only when ς_i is an integer and $l_i = 0$ so that the logarithmic term vanishes. Note that the space of locally holomorphic solutions around s_i could be at most two-dimensional. If that happens at a point s_i with $1 < i \leq m$, then a local holomorphic solution around point s_{i-1} can always be decomposed as linear combination (19) because ϕ_{i1} and ϕ_{i2} is the basis of solutions, so the constants ζ_i and ξ_i are fixed. This way the two local expansions φ_{i-1} and φ_i coincide on U_i and there are no additional conditions at this point.

Similarly, if there were two solutions at s_i , hence at all s_j for $j < i$, but only one holomorphic solution at s_{i+1} , then this single solution can always be decomposed into φ_i as in (19). This fixes the constants ζ_i and ξ_i and we proceed with this solution to the next singular point. Note that this will also fix all the previous constants ζ_j and ξ_j because we are left with only one solution.

Finally, if there is only a single solution φ_i at s_i and likewise φ_{i+1} at s_{i+1} , they must be linearly dependent in order for them to be expansions of the same entire function. So, there exist $\mathbb{C} \ni (\alpha_i, \beta_i) \neq (0, 0)$, such that

$$F_i(z) := \alpha_i \varphi_i(z) + \beta_i \varphi_{i+1}(z) = 0 \quad \text{for all } z \in U_i. \quad (20)$$

As $F_i(z)$ vanishes on a non-empty open set on which it is

holomorphic, it vanishes identically. Thus we also have

$$F'_i(z) := \alpha_i \varphi'_i(z) + \beta_i \varphi'_{i+1}(z) = 0 \quad \text{for all } z \in U_i. \quad (21)$$

From (20) and (21) we deduce

$$W_i(z) := \det \begin{bmatrix} \varphi_i(z) & \varphi_{i+1}(z) \\ \varphi'_i(z) & \varphi'_{i+1}(z) \end{bmatrix} = 0, \quad \text{for all } z \in U_i.$$

But $W_i(z)$ is the Wronskian of two solutions of the same equation, so if it vanishes at one point $z_i \in U_i$, then it vanishes identically on U_i . Proceeding this way we obtain at most $m - 1$ “gluing” conditions $W_i(z_i) = 0$ for $1 \leq i < m$, which guarantee that $\varphi(z)$ is an entire function.

However, functions distinguished by these conditions are not necessarily elements of \mathcal{H} . We also need to check if $\langle \varphi, \varphi \rangle < \infty$. As mentioned above it depends primarily on the order and type of the entire functions. If infinity is a regular point, then necessarily $\varphi(z) \in \mathcal{H}$, because the growth is subexponential.

In general, if the infinity is an irregular singular point the problem is really hard especially when the considered equation is of order higher than two. One has to determine asymptotic expansions of solutions at infinity. In the simplest case, which holds for the two models considered here, they are of the form

$$\varphi \sim e^{\sigma z^e} z^\rho \left(1 + \mathcal{O}\left(\frac{1}{z}\right) \right). \quad (22)$$

These series are formal, but they give bounds for the growth order ϱ of the function, and its type σ [32].

To investigate these problems in whole generality quite involved mathematical techniques must be used. So, the whole exposition of more general version of our method will be published separately.

APPLICATION TO THE FIRST MODEL

Our method applied to the generalized Rabi model (8) provides a closed-form formula for the spectrum of the problem. It is given as zeros of a certain transcendental function $W(p)$ expressed in terms of the confluent Heun functions. Here $p := (x, \lambda, \mu, \epsilon)$, $x := E + \lambda^2$ is taken as a spectral parameter. System (8) written as the second order equation (13) after change of variables

$$v(y) := \exp(-2\lambda^2 y) \psi_1(\lambda(2y - 1)), \quad z = \lambda(2y - 1),$$

transforms to the confluent Heun equation

$$v'' + \left(\alpha + \frac{\beta + 1}{y} + \frac{\gamma + 1}{y - 1} \right) v' + \left(\frac{\tilde{\mu}}{y} + \frac{\tilde{\nu}}{y - 1} \right) v = 0, \quad (23)$$

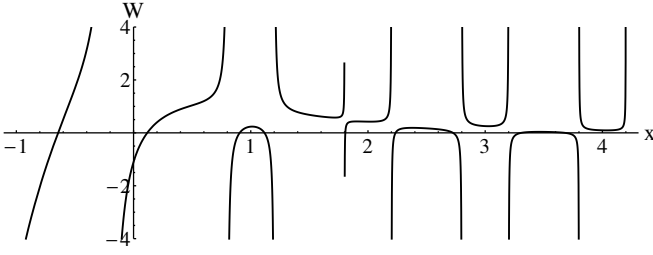


FIG. 1. Graph of Wronskian $W(p)$ for $p := (x, \lambda, \mu, \epsilon) = (x, 4/10, 7/10, 1/5)$.

where

$$\begin{aligned}\tilde{\mu} &= \frac{1}{2}(\alpha - \beta - \gamma + \alpha\beta - \beta\gamma) - \eta, \\ \tilde{\nu} &= \frac{1}{2}(\alpha + \beta + \gamma + \alpha\gamma + \beta\gamma) + \delta + \eta.\end{aligned}\quad (24)$$

In terms of $(x, \lambda, \mu, \epsilon)$ the above parameters are given by

$$\begin{aligned}\alpha &= 4\lambda^2, & \beta &= -x + \epsilon, \\ \gamma &= -1 - x - \epsilon, & \delta &= 2(1 - 2\epsilon)\lambda^2, \\ 2\eta &= 1 - 2\mu^2 + (1 + x)(x - 4\lambda^2) + \epsilon(1 + 4\lambda^2) - \epsilon^2.\end{aligned}\quad (25)$$

The characteristic exponents at regular singularities $y = 0$ and $y = 1$ are $\{0, -\beta\}$ and $\{0, -\gamma\}$, respectively.

If $v_1(p; y)$ and $v_2(p; y)$ are two solutions of (23), then their Wronskian is defined in usual way

$$w[v_1, v_2](p; y) := v_1'(p; y)v_2(p; y) - v_1(p; y)v_2'(p; y). \quad (26)$$

But to simplify notation, to denote this Wronskian we just write $w(p; y)$ if solutions v_1 and v_2 are specified.

Solutions of Eq. (23) are the confluent Heun functions, see [34],

$$H_1(y) := \text{HeunC}(a_0; y), \quad H_2(y) := \text{HeunC}(a_1; 1 - y), \quad (27)$$

with parameter sets $a_0 := (\alpha, \beta, \gamma, \delta, \eta)$, and $a_1 := (-\alpha, \gamma, \beta, -\delta, \delta + \eta)$. The continuation condition is given by their Wronskian

$$w(p; y) := H_1(y)H_2'(y) - H_1'(y)H_2(y), \quad (28)$$

in the following way $W(p) := w(p; 1/2) = 0$.

In the above formulae we assumed that neither $-\gamma = 1 + x + \epsilon$, nor $-\beta = x - \epsilon$ is a non-negative integer. This is the generic case, where only one characteristic exponent around each point is integer (equal to zero). This only leaves out at most a finite number of spectrum points, as described below.

The graph of $W(p)$ and the spectrum for $\epsilon = 0.2$ are shown in Figs. 1 and 2, respectively.

For comparison, we show the graph of $W(p)$ as a func-

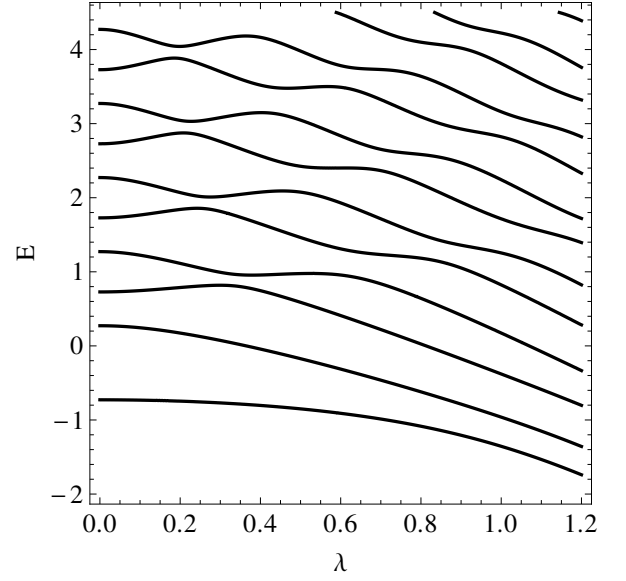


FIG. 2. Spectrum of generalized Rabi model for $\mu = 0.7$, and $\epsilon = 0.2$.

tion of x for the Rabi model (i.e., for $\epsilon = 0$) in Fig. 3. The corresponding spectrum is shown in Fig. 4.

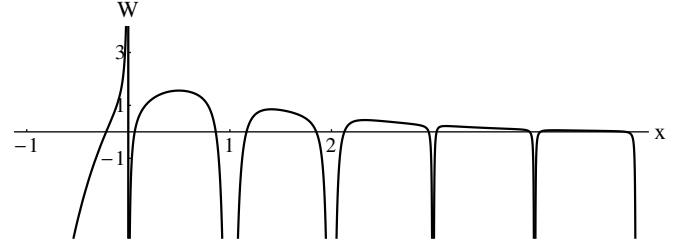


FIG. 3. Graph of Wronskian $W(p)$ for $p := (E, \lambda, \mu, \epsilon) = (E, 7/10, 4/10, 0)$.

When one of the other characteristic exponents is a natural number, the Heun function might involve a logarithm but then there always exists a local holomorphic solution corresponding to the larger exponent. These solutions are given by

$$\begin{aligned}H_3(y) &:= y^{-\beta}(y - 1)^{-\gamma}\text{HeunC}(c_0, y), \\ H_4(y) &:= (y - 1)^{-\gamma}\text{HeunC}(c_1, 1 - y),\end{aligned}\quad (29)$$

where $c_0 := (\alpha, -\beta, -\gamma, \delta, \eta)$ and $c_1 := (-\alpha, -\gamma, \beta, -\delta, \delta + \eta)$. Additionally, the logarithmic term can vanish, making both expansions at a given point locally holomorphic. This happens when the parameters satisfy an additional constraint which coincides with the Δ_n condition by Fiziev, who showed that the confluent Heun function is then just a polynomial [35]. Since both of these options could arise at any of the two points, let us introduce the

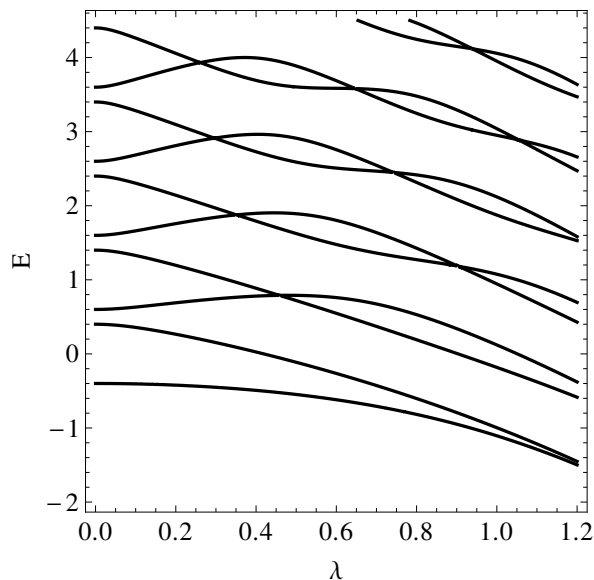


FIG. 4. Spectrum of the Rabi model for $\mu = 0.4$.

following notation, ς_s will denote the exponent in question at a point $s \in \{0, 1\}$, and $\Delta_s := \Delta_{\varsigma_s}(c_s)$. Keep in mind that the Δ condition depends on parameters differently at each point and also that it is given recursively and the integer exponent ς_s specifies how far the recurrence is carried. We then end up with three possible sub-cases. For each of them we have to specify two solutions for calculations of Wronskian.

1. At one singular point $s \in \{0, 1\}$ both exponents are integer and there are logarithmic terms in local solutions, i.e. $\varsigma_s \in \mathbb{N}$ and $\Delta_s \neq 0$. Moreover, at the other singular point only one exponent is an integer, i.e. $\varsigma_{1-s} \notin \mathbb{N}$. In this case we take H_{s+3} and H_{2-s} as two solutions for calculation of the Wronskian.
2. If, for both $s \in \{0, 1\}$, $\varsigma_s \in \mathbb{N}$ and $\Delta_s \neq 0$, then take H_3 and H_4 . Notice that in this case, logarithmic terms in local solutions around each point appear.
3. If for one point $s \in \{0, 1\}$ we have $\varsigma_s \in \mathbb{N}$, $\Delta_s = 0$, then both local solutions around this point are holomorphic. One can show that, in this case, Heun equation (25) has an entire solution so the continuation condition is fulfilled.

The third subcase is rather remarkable because the vanishing of the logarithmic term gives us two locally holomorphic solutions around one point, and there is always at least one good solution around the second point—either one corresponding to the zero exponent, or the other corresponding to the second integer exponent. A locally holomorphic solution around the second singular point must be a linear combination of the two local solutions around the first singular point, as stated in the Method

section. As it can be continued throughout a set containing both singular points, its radius of convergence is larger than the distance between those points, so it must be infinite.

The sets of parameter values for each Δ condition can intersect, meaning that there could be two integer exponents, which in turn give two points of spectrum $x_1 = \epsilon - \beta$ and $x_2 = -\epsilon - \gamma - 1$, each with one entire eigenstate. It is further possible that the x_i coincide giving a degenerate energy level with two entire eigenstates, and since both exponents are integers it follows that in such a case necessarily $2\epsilon \in \mathbb{Z}$. This is the most “degenerate” case, when both logarithmic terms vanish. Additionally, if the exponents differ by one, i.e. $\beta = \gamma + 1$, the above implies $\epsilon = 0$ and we recover the classical Judd states of the unperturbed Rabi model described, e.g., in [19, 36].

Finally, concerning the finiteness of the norm, we observe that for differential equation (13) with $p(z)$ and $q(z)$ given by (14), the Poincaré rank is 1, and the asymptotic expansions of its solutions are

$$\begin{aligned}\varphi_1(z) &\sim e^{\lambda z} z^{E+\lambda^2+\epsilon-1} \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right), \\ \varphi_2(z) &\sim e^{-\lambda z} z^{E+\lambda^2-\epsilon} \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right),\end{aligned}\tag{30}$$

as can be checked by direct substitution. Here, $\varrho \leq 1$, so we conclude that all entire solutions of this equation belong to \mathcal{H} .

With $\epsilon = 0$ system (8) has a \mathbb{Z}_2 symmetry. It is invariant with respect to the involution $\tau : \mathcal{H}^2 \rightarrow \mathcal{H}^2$ given by $\tau(\psi_1, \psi_2)(z) = (\psi_2(-z), \psi_1(-z))$. In other words, if $(\psi_1(z), \psi_2(z))$ is a solution of this system, then also $(\psi_2(-z), \psi_1(-z))$ is its solution. We say that a solution $\psi = (\psi_1, \psi_2)$ of (8) has parity $\sigma \in \{-1, +1\}$, if $\tau(\psi) = \sigma\psi$.

Analyzing our method of determination of the spectrum for the Rabi model we noticed several important facts. First of all we asked if the discrete symmetry of the Rabi problem, whose role was so strongly underlined in [7], is really important for determination of the spectrum. Our answer to this question is negative. Amazingly enough, its explicit use in the Rabi model, hides somehow a good way to attack the problem for which its analytical nature plays the crucial role. A necessary condition for $\psi(z)$ to be an eigenvector is that it must be holomorphic in the whole complex plane. Here it is worth to mention that this is only a necessary condition, not necessary and sufficient one. This fact is of crucial importance for a proper physical interpretation of the obtained results. Simply, we can mistakenly interpret certain values of energy as eigenvalues of the Hamiltonian.

APPLICATION TO THE SECOND MODEL

The general procedure of finding an entire solution is the same for a system as for a single second order equation. We look for a holomorphic solution around each singular point and then we “glue” them together to one entire solution in that the local solutions around two singular points must coincide. The technical difference is that each solution is a vector, so the local series is determined by a matrix recurrence.

System (11) has two regular singular points $s \in \{-1, +1\}$, which are poles of the matrix of coefficients $\mathbf{A}(y)$. We look for local solution around these points that have the form

$$F(\rho, y) = (y - s)^\rho \sum_{n=0}^{\infty} (y - s)^n a_n(\rho), \quad (31)$$

where, $F = [F_1, F_2]^T$, and $a_n = [a_{n,1}, a_{n,2}]^T \in \mathbb{C}^2$. To determine ρ and a_0 we substitute the above series into system (11), and require that the lowest order term vanishes:

$$(\mathbf{A}_s - \rho \text{Id}_2) a_0 = 0, \quad (32)$$

where \mathbf{A}_s is the residue matrix of $\mathbf{A}(y)$ at $y = s$. In other words, ρ must be an eigenvalue of \mathbf{A}_s and a_0 the corresponding eigenvector. In our case, the poles of $\mathbf{A}(y)$ are all simple, so the eigenvalues will be the characteristic exponents. For both points we simply have $\rho \in \{0, x\}$, where

$$x = \frac{4g^2 + 4\omega E + \omega_0 U}{4\omega^2 - U^2}, \quad (33)$$

and the nonzero exponent is the same spectral parameter as for the basic Rabi model, i.e. when $U = 0$, $\omega = 1$ and $g = \lambda$.

Once a_0 is determined, the other coefficients are given by a recurrence relation of the form

$$(\mathbf{A}_s - (\rho + n) \text{Id}_2) a_n = R(a_{n-1}, \dots, a_0), \quad (34)$$

for some linear function R . If x is not an integer, then at both singular points there are solutions of the form (31) with each of the exponents. However, the only solution that can be entire is the one with $\rho = 0$ which we denote by $\psi^{(s,0)}(y) := F(0, y)$.

A local solution around the second singular point can be found in a similar way. However, we can obtain it in a simpler way. Let us notice that system (11) has the following symmetry. If $\psi(y)$ is its solution, then

$$\tilde{\psi}(y) := \sigma_z \psi(-y), \quad (35)$$

is also its solution. Hence

$$\tilde{\psi}^{(s,0)}(y) := \sigma_z \psi^{(s,0)}(-y), \quad (36)$$

is a solution of (11). But $\tilde{\psi}(y)$ is a power series in $(y + s)$, so it is a local holomorphic solution around the other singular point. In other words

$$\psi^{(-s,0)}(y) := \sigma_z \psi^{(s,0)}(-y). \quad (37)$$

Note that this only gives advantage in calculations because only one series has to be determined, but the existence of such a symmetry does not influence the fundamental conditions to be met.

If the above two solutions coincide in their common domain of definition, then their Wronskian

$$w(p, y) := \det [\psi^{(s,0)}(y), \sigma_z \psi^{(s,0)}(-y)], \quad (38)$$

vanishes for arbitrary y . In this model $p := (E, \omega, \omega_0, g, U)$. Exemplary spectrum obtained from condition $W(p) := w(p, 0) = 0$ for $\omega = 2\omega_0 = -U = 2$ is shown in Fig. 5.

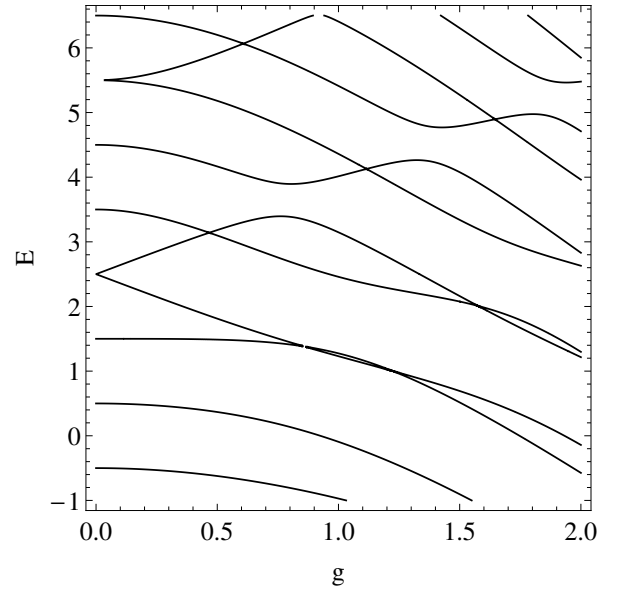


FIG. 5. Spectrum of the the second generalized Rabi model for $\omega_0 = 1$, $\omega = 2$ and $U = -2$.

When x is a positive integer, say m , there might arise logarithmic terms in the solution, and the form (31) only works for one of the exponents. An elegant way to recover both solutions, due to Frobenius, is to take the series (31)

with different, rescaled $a_0(\rho)$ and define solutions as

$$\begin{aligned}\psi^{(s,x)}(y) &:= F(0, y), \\ \psi^{(s,l)}(y) &:= \left. \frac{\partial F(\rho, y)}{\partial \rho} \right|_{\rho=0} \\ &= \ln(y-s)F(0, y) + \sum_{n=0}^{\infty} a'_n(0)(y-s)^n.\end{aligned}\quad (39)$$

The choice of $a_0(\rho)$ must be such that the first $m-1$ terms of F vanish at $\rho=0$, and the first solution actually corresponds to the higher integer exponent $x=m$. For a detailed exposition see [32].

The determination of the entire solutions here actually unites the Juddian solutions and the infinite family of non-polynomial ones discovered in [19]. To wit, the series $\psi^{(s,x)}$ is always locally analytic, so that it enters into the usual Wronskian condition of connecting solutions around different points. If this Wronskian vanishes, then there exists one entire solution, regardless of the presence of the logarithmic term.

By construction, the second solution $\psi^{(s,l)}$ is such that the second series is well defined for all values of parameters, so any condition that the logarithmic term vanishes must be given by a common factor J_m of all the a_n entering the first series $F(0, y)$. In such a case, the corresponding coefficient of $\psi^{(-s,x)}(y)$, which is $\sigma_z a_m$, will also contain that factor. The Wronskian condition will be

$$\det [\psi^{(s,x)}(y), \sigma_z \psi^{(s,x)}(-y)] = 0, \quad (40)$$

and it will include the condition $J_m = 0$ because both vectors are proportional to it. At the same time it will contain the condition for the solution to be entire in case the logarithm does not vanish because the two series with exponent $x=m$ around different points will coincide and define an entire function.

The solutions with $J_m = 0$ are the counterparts of the classical Juddian solutions, and they appear when the parameters lie on some algebraic curves in the parameter space. Recall that the standard Juddian curves were m ovals, restricted to a finite region of g and ω_0 and also that for $\omega_0 = 0$ the system decoupled trivially. Here, each J_m has a factor which corresponds to a deformation of the $\omega_0 = 0$ case, given by the following parabola in the (ω_0, g) plane

$$\omega_0 = \frac{4g^2 U}{4\omega^2 - U^2} - Ux. \quad (41)$$

This can be verified by direct substitution into the **A** matrix, which becomes diagonal, so that the system decouples and can be solved explicitly

$$\psi_{1,2} = \exp\left(\mp \frac{4g^2}{4\omega^2 - U^2} y\right) (y \pm 1)^x. \quad (42)$$

Remarkably (41) is the only Juddian condition that can be given explicitly for any integer x , in contrast to the other conditions, which have to be determined recursively. Obviously it is not confined like the usual ovals and gives nontrivial parameter values for arbitrarily large g . The other conditions are much more involved than in the classical model, e.g., the first one corresponding to $x=m=1$ is

$$\begin{aligned}J_1 &= (4g^2 U + (U^2 - 4\omega^2)(U + \omega_0)) \\ &\quad (16g^4 U^2 + (U^2 - 4\omega^2)^2((U + \omega_0)^2 - 4\omega^2) + \\ &\quad 8g^2(U^2 - 4\omega^2)(U(U + \omega_0) - 8\omega^2)),\end{aligned}\quad (43)$$

where the first factor is the aforementioned parabola. Note that when the parameters are chosen so that $J_m = 0$ the series $F(0, y)$ vanishes, so the actual solution with the higher exponent is \tilde{F} such that $F(0, y) = J_m \tilde{F}$.

Similarly to the previous model, thanks to the additional coupling, it is possible for the closed curves defined by J_m to intersect the parabola (41) of some J_n in which case there will be two integer values of x in the spectrum, which is a novel feature of the model.

Regarding the normalization condition, for the system (11), one has the expansions

$$\psi_{\pm}(z) \sim e^{\pm \sigma z} z^v \left[\frac{\sqrt{2\omega + U} + \mathcal{O}\left(\frac{1}{z}\right)}{\mp \sqrt{2\omega - U} + \mathcal{O}\left(\frac{1}{z}\right)} \right], \quad (44)$$

with

$$\sigma = \frac{2g}{\sqrt{4\omega^2 - U^2}}, \quad v = \frac{4g^2 + 4\omega E + \omega_0 U}{4\omega^2 - U^2}, \quad (45)$$

so that the growth order is one, and all entire solutions belong to \mathcal{H} .

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[1] I. I. Rabi, On the process of space quantization, Phys. Rev. 49 (1936) 324–328.

[2] G. Günter, A. A. Anappara, J. Hees, A. Sell, G. Biaisol, L. Sorba, S. De Liberato, C. Ciuti, A. Tredicucci, A. Leitenstorfer, R. Huber, Sub-cycle switch-on of ultra-strong light-matter interaction, Nature 458 (7235) (2009) 178–181.

- [3] T. Niemczyk, F. Deppe, H. Huebl, E. P. Menzel, F. Hocke, M. J. Schwarz, J. J. García-Ripoll, D. Zueco, T. Hümmer, E. Solano, A. Marx, R. Gross, Circuit quantum electrodynamics in the ultrastrong-coupling regime, *Nature Physics* 6 (2010) 772–776.
- [4] P. Forn-Díaz, J. Lisenfeld, D. Marcos, J. J. García-Ripoll, E. Solano, C. J. P. M. Harmans, J. E. Mooij, Observation of the Bloch-Siegert Shift in a Qubit-Oscillator System in the Ultrastrong Coupling Regime, *Phys. Rev. Lett.* 105 (23) (2010) 237001.
- [5] A. Crespi, S. Longhi, R. Osellame, Photonic Realization of the Quantum Rabi Model, *Phys. Rev. Lett.* 108 (16) (2012) 163601.
- [6] J. Casanova, G. Romero, I. Lizuain, J. J. García-Ripoll, E. Solano, Deep Strong Coupling Regime of the Jaynes-Cummings Model, *Phys. Rev. Lett.* 105 (26) (2010) 263603.
- [7] D. Braak, Integrability of the Rabi model, *Phys. Rev. Lett.* 107 (2011) 100401.
- [8] A. L. Grimsmo, S. Parkins, Cavity-QED simulation of qubit-oscillator dynamics in the ultrastrong-coupling regime, *Phys. Rev. A* 87 (2013) 033814.
- [9] A. L. Grimsmo, S. Parkins, Open Rabi model with ultrastrong coupling plus large dispersive-type nonlinearity: Nonclassical light via a tailored degeneracy, *Phys. Rev. A* 89 (2014) 033802.
- [10] I. D. Feranchuk, L. I. Komarov, A. P. Ulyanenko, Two-level system in a one-mode quantum field: numerical solution on the basis of the operator method, *J. Phys. A: Math. Gen.* 29 (1996) 4035–4047.
- [11] C. Durst, E. Sigmund, P. Reineker, A. Scheuing, Treatment of non-adiabatic Hamiltonians by matrix continued fractions. I. Electronic two-level system coupled to a single vibrational mode, *J. Phys. C: Solid State Phys.* 19 (1986) 2701–2720.
- [12] É. A. Tur, Energy Spectrum of the Hamiltonian of the Jaynes-Cummings Model without Rotating-Wave Approximation, *Opt. Spectrosc.* 91 (2001) 899–902.
- [13] C. Emary, R. F. Bishop, Bogoliubov transformations and exact isolated solutions for simple nonadiabatic Hamiltonians, *J. Math. Phys.* 43 (2002) 3916–3926.
- [14] D. Braak, Continued Fractions and the Rabi Model, *J. Phys. A: Math. Theor.* 46 (2013) 175301.
- [15] D. Braak, A generalized g -function for the quantum Rabi model, *Ann. Phys. (Berlin)* 525 (3) (2013) L23–L28.
- [16] A. Moroz, On the spectrum of a class of quantum models, *EPL* 100 (6) (2012) 60010.
- [17] Q.-H. Chen, C. Wan, S. He, T. Liu, K.-L. Wang, Exact solvability of the quantum Rabi model using Bogoliubov operators, *Phys. Rev. A* 86 (2012) 023822.
- [18] A. J. Maciejewski, M. Przybylska, T. Stachowiak, How to calculate spectra of Rabi and related models, *ArXiv e-prints* 1210.1130 [math-ph].
- [19] A. J. Maciejewski, M. Przybylska, T. Stachowiak, Full spectrum of the Rabi model, *Phys. Lett. A* 378 (1-2) (2014) 16–20.
- [20] V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform, *Comm. Pure Appl. Math.* 14 (1961) 187–214.
- [21] D. J. Newman, H. S. Shapiro, A Hilbert space of entire functions related to the operational calculus, mimeographed notes, University of Michigan, Ann Arbor, 1964, 92 pp.
- [22] D. J. Newman, H. S. Shapiro, Certain Hilbert spaces of entire functions, *Bull. Amer. Math. Soc.* 72 (1966) 971–977.
- [23] E. Fischer, Über algebraische Modulsysteme und lineare homogene partielle Differentialgleichungen mit konstanten Koeffizienten, *J. für Math.* 140 (1911) 48–81.
- [24] E. Fischer, Über die differentiationsprozesse der Algebra, *J. für Math.* 148 (1917) 1–78.
- [25] B. Y. Levin, Lectures on entire functions, Vol. 150 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI, 1996.
- [26] R. P. Boas, Jr., Entire functions, Academic Press Inc., New York, 1954.
- [27] A. Vourdas, Analytic representations in quantum mechanics, *J. Phys. A, Math. Gen.* 39 (2006) R65.
- [28] D. Braak, Solution of the Dicke model for $n = 3$, *J. Phys. B: At. Mol. Opt. Phys.* 46 (22) (2013) 224007.
- [29] I. Travěnek, Solvability of the two-photon Rabi Hamiltonian, *Phys. Rev. A* 85 (2012) 043805.
- [30] M. Kuś, A. J. Maciejewski, M. Przybylska, T. Stachowiak, Solvability of multi-level Rabi systems, in preparation.
- [31] A. J. Maciejewski, M. Przybylska, T. Stachowiak, An exactly solvable system from quantum optics, *Phys. Lett. A*, submitted for publication.
- [32] M. Kohn, Global analysis in linear differential equations, Vol. 471 of *Mathematics and its Applications*, Kluwer Academic Publishers, Dordrecht, 1999.
- [33] J. B. Conway, Functions of one complex variable, 2nd Edition, Vol. 11 of *Graduate Texts in Mathematics*, Springer-Verlag, New York-Berlin, 1978.
- [34] A. Ronveaux, Heun’s Differential Equations, Oxford University Press, Oxford, 1995.
- [35] P. P. Fiziev, Novel relations and new properties of confluent Heun’s functions and their derivatives of arbitrary order, *J. Phys. A* 43 (3) (2010) 035203.
- [36] M. Kuś, M. Lewenstein, Exact isolated solutions for the class of quantum optical systems, *J. Phys. A* 19 (2) (1986) 305–318.